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MAXIMAL ELEMENTS IN COMPACT SEMIGROUPS

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ABSTRACT

In this paper we investigate the structure of compact connected semigroups satisfying $S^2 = S$ by studying the properties of maximal elements in S .

In chapter 1 it is shown that an \mathcal{L} -maximal idempotent in such a semigroup S with a zero cannot be a weak cutpoint. Consequently if an \mathcal{L} -maximal element is a weak cutpoint, then $S = K$. Our attention is next focused on the local separating properties of maximal elements in compact connected semigroups. It is first shown that an \mathcal{L} -maximal element contained in SE but not in K cannot be a local separating point of S . Thus if a \mathcal{J} -maximal element of S is a local separating point of S , then $S = K$. If an \mathcal{L} -maximal element p of S separates S locally and p is not contained in SE , then p is a local separating point of $eS \cup fS$ for some idempotents e and f in S . Finally by using these results, the local separating properties of maximal \mathcal{J} -classes are studied. It is shown that \mathcal{J}_e is not a local separating set of S if $S = SeS$. The same conclusion holds under the hypotheses $S^2 = S$ and e is a \mathcal{J} -maximal idempotent providing that H_e is totally disconnected.

In chapter 2 we consider a class of "finitely floored" spaces: such a space admits the structure of a semigroup with $S = ESE$ only if $S = K$. In particular we show that if $S = ESE$ and has a zero, then for any non-zero element h in $H^2(S)$ there exist a pair of idempotents e and f in S such that $h|Se \cup Sf \neq 0$. Thus S cannot be floor for h , for then S would equal $Se \cup Sf$ which cannot happen. Using this

idea and the concept of "peripherality," we show that if S is a continuum with $S = ESE$ and S is a floor for a finite subset of $H^2(S)$, then $S = K$. An interesting consequence of these techniques is that a maximal idempotent cannot lie in the unique floor of an element of $H^2(S)$.

Next in chapter 3 the results of chapters 1 and 2 are applied to give conditions under which maximal idempotents in semigroups with $S = ESE$ fail to have 2-dimensional Euclidean neighborhoods. In particular it is shown that a \mathcal{J} -maximal idempotent e cannot have a 2-dimensional Euclidean neighborhood if \mathcal{J}_e is totally disconnected. Now using the local separating properties of maximal \mathcal{J} -classes, we have that \mathcal{J}_e is totally disconnected if either H_e is totally disconnected or $S = SeS$. In either of these cases e does not have a 2-dimensional Euclidean neighborhood. As a corollary it follows that if $S = ESE$ is a subcontinuum of the plane, R^2 , then each maximal idempotent lies on the boundary of S in R^2 .

INTRODUCTION

In the study of topological semigroups one of the fundamental questions is "given a space, does it admit the structure of a topological semigroup satisfying property (*)?" Property (*) is any algebraic condition which one would like to impose. In this paper we shall consider a variation on this general question. Namely, given a semigroup S satisfying property (*), is it necessarily the case that $S = K$, K being the minimal ideal. Here property (*) is to be any topological condition as well as algebraic which one imposes on S . This type of question has been considered earlier by Koch and Wallace [1957] and Cohen and Koch [1965]. Throughout this paper property (*) will include (i) $S^2 = S$ and (ii) S is compact and connected.

In a compact semigroup S satisfying $S^2 = S$, maximal idempotents exist outside each ideal of S . Thus a natural approach to the general problem is to determine the local topological structure of S at the maximal idempotents. The best result here, in the case that S has an identity element e , is that e must be peripheral [Hofmann and Mostert; 1966].

In chapter 1 we adopt this approach by investigating the local separating properties of maximal idempotents in such semigroups. We show that a maximal idempotent in S cannot be a weak cutpoint; if S is a continuum such that every point is

a weak cutpoint, then S must equal K . It is also shown that maximal elements in a semigroup S satisfying $S^2 = S$ cannot be local cutpoints, and using this fact we give conditions under which maximal \mathcal{J} -classes fail to be local separating sets.

Our next approach, in chapter 2, is to make use of the cohomology theory of compact semigroups to study the global structure of semigroups where it is assumed that $S = ESE$. Here it is shown that if S is a semigroup on a "finitely floored space" satisfying $S = ESE$, then $S = K$. By making use of these techniques, it follows that a maximal idempotent cannot lie on the unique floor of some element of $H^2(S)$.

In chapter 3 we make use of the results of the preceding chapters to give conditions under which maximal idempotents fail to have 2-dimensional Euclidean neighborhoods. It follows as a corollary of this that if S is a compact connected 2-manifold without boundary satisfying $S = ESE$, then $S = K$ [Cohen and Koch; 1965].

CHAPTER 0

PRELIMINARIES

The purpose of this chapter is to establish notation and to agree upon terminology. Some of the more frequently used results concerning topological semigroups are given, with proofs only when it is necessary and seems practical. As it is impossible to give a complete bibliography for this paper, a useful reference for the reader is Hofmann and Mostert [1966].

Topological Preliminaries. A space X is to mean a Hausdorff topological space; let it be emphasized that all spaces are Hausdorff. If A and B are subsets of a space X , then $A \setminus B$ denotes the complement of B in A . The empty set is denoted by \square . The closure of a subset A in a space X is denoted by A^* ; its interior, by A° . A is said to be nowhere dense if $(A^*)^\circ = \square$. The boundary of A , written as $F(A)$, is the set of all points which are interior to neither A nor $X \setminus A$. It is immediate that $F(A) = A^* \cap (X \setminus A)^*$.

A continuum is a compact connected space. The following theorem, due to Janiszewski [Kuratowski; 1948], is of importance in the sequel.

0.1 Theorem. Let X be a continuum and Y a subcontinuum of X . If A is a subset of X not containing Y , C a component of $Y \cap A$, then $C^* \cap F(A) \neq \square$.

Let X and Y be spaces. A relation from X to Y is a subset of the cartesian product $X \times Y$. If R is a closed subset of $X \times Y$, then R is a closed relation from X to Y . A subset of $X \times X$ is called a relation on X . If R is a relation from X to Y , it is convenient to write xRy to denote that $(x,y) \in R$. If R is an equivalence relation on X (i.e., R is reflexive, antisymmetric, and transitive), R_x denotes the equivalence class of R containing x ; thus $R_x = \{y \in X: (x,y) \in R\}$. For an equivalence relation R on a space X , X/R denotes the set of equivalence classes of R . If X is compact and R is closed, X/R endowed with the quotient topology is a space. Now let A be a closed subset of the space X . Define R_A to be the set of all pairs (x,y) in $X \times X$ such that either $x = y$ or x and y are elements of A . Then the space X/R_A is written X/A and can be thought of as the space obtained from X by shrinking A to a point.

Suppose that \leq is a relation from X to Y . For $y \in Y$, $L(y)$ is defined to be the set of elements x in X for which $x \leq y$. If A is a subset of Y , then $L(A)$ denotes the union of all sets $L(a)$ for which $a \in A$. It is easily verified that if Y is compact and \leq is closed, $L(A)$ is closed for each closed subset of A of Y .

A net in a space X is a function from a directed set D into X . For convenience a net $\theta: D \rightarrow X$ is denoted simply by (x_α) , where the domain is understood to be some directed set D and x_α denotes the image of α under θ for each $\alpha \in D$. If D is a directed set (directed by \leq) and A is a subset of D , then A is cofinal (residual) in D if for each $\alpha \in D$, there exists

$\gamma \in D$ such that $\alpha \leq \gamma$ and $\gamma \in A$ (and for each $\beta \in D$ with $\gamma \leq \beta$, $\beta \in A$).

Let 2^X denote the space of closed subsets of the space X . If (A_α) is a net in 2^X with domain D , its limit superior, denoted by $\limsup A_\alpha$, is the set of all $x \in X$ such that for each open set U containing x , $\{\alpha \in D : A_\alpha \cap U \neq \emptyset\}$ is cofinal in D . Its limit inferior, denoted by $\liminf A_\alpha$, is the set of all $x \in X$ such that for any open set U containing x , $\{\alpha \in D : U \cap A_\alpha \neq \emptyset\}$ is residual in D . If $\limsup A_\alpha = \liminf A_\alpha = A$, the net (A_α) is said to converge to A , and we write $\lim A_\alpha = A$.

The cohomology used in this paper is Alexander-Wallace-Spanier theory with coefficient group arbitrary unless specified.

A point p in a compact space X is peripheral if for each open U containing p , there exists an open set V containing p which is contained in U , and such that the natural induced homomorphism $H^n(X) \xrightarrow{i^*} H^n(X \setminus V)$ is an isomorphism for all $n \geq 0$. That is to say, there exists small open sets V containing p , such that $H^n(X, X \setminus V) = 0$ for all $n \geq 0$. By the map excision theorem [Wallace; 1952] it follows that this last assertion is equivalent to $H^n(V^*, F(V)) = 0$ for all $n \geq 0$. It now follows easily that if A is a closed subspace of X and $p \in A^0$, then p is peripheral in X if and only if p is peripheral in A . Thus the concept of peripheralness in this sense, is seen to be a local one. The following proposition is given here for later use.

0.2 Proposition. If X is a continuum and p is peripheral in X , then X is 1-semi-locally connected at p .

Topological Algebra Preliminaries. A topological semigroup is a space endowed with a continuous associative multiplication. To be precise, a topological semigroup is a pair (S, m) , where S is a space and m is a continuous function from $S \times S$ into S satisfying $m(x, m(y, z)) = m(m(x, y), z)$. If no confusion seems likely, we simply say that S is a semigroup and denote $m(x, y)$ by xy . Without exception a semigroup is to mean a topological semigroup, and S is used to denote a semigroup.

A non-empty subset A of S is a left (right) ideal if $SA \subseteq A$ ($AS \subseteq A$). A two-sided ideal (or ideal) is both a left and a right ideal. If A is a closed ideal in S , then S/A , the Rees quotient module A , is in a natural manner a semigroup with a zero. If S has a minimal ideal, it is denoted by K and called the kernel of S . If S is compact, then the minimal ideal K exists and is known to be a retract [Wallace; 1957]. For further information concerning the structure of K the reader is referred to Clifford and Preston [1961] and Wallace [1956].

If S is a semigroup and $a \in S$, the smallest left (right, two sided) ideal containing a is denoted by $L(a)$ ($R(a)$, $J(a)$). Clearly we have the following identities ($\{a\}$ is replaced by a when no confusion seems likely) :

$$L(a) = a \cup Sa$$

$$R(a) = a \cup aS$$

$$J(a) = a \cup Sa \cup aS \cup SaS.$$

Green's relations are then defined on S as follows: for $x, y \in S$

$$\begin{aligned}
x \leq (\mathcal{L}) y & \quad \text{if } L(x) \subseteq L(y) \\
x \leq (\mathcal{R}) y & \quad \text{if } R(x) \subseteq R(y) \\
x \leq (\mathcal{J}) y & \quad \text{if } J(x) \subseteq J(y) \\
x \leq (\mathcal{K}) y & \quad \text{if } R(x) \subseteq R(y) \text{ and } L(x) \subseteq L(y)
\end{aligned}$$

The relations \leq so defined are quasi-orders and each is closed.

To each there is associated an equivalence relation \mathcal{L} , \mathcal{R} , \mathcal{J} , or \mathcal{K} , respectively, defined by

$$x \mathcal{L} y \quad \text{if} \quad x \leq (\mathcal{L}) y \quad \text{and} \quad y \leq (\mathcal{L}) x, \text{ etc.}$$

An element e in a semigroup S is an idempotent if $e^2 = e$.

The set of idempotents of S is denoted by E . The following theorems are stated for convenience of the reader [Wallace; 1953a], [Koch; 1957a].

0.3 Theorem. Let S be a compact semigroup and A a closed subset of S . If $A \subseteq Ax$ for some $x \in S$, then $A = Ax$.

0.4 Theorem. Let S be compact and $a \in S$. If $a \in Sa$ ($a \in SaS$), then $a \in ES$ ($a \in ESE$).

A compact semigroup S with identity e is irreducible if it is connected and does not contain a proper compact connected subsemigroup T with $e \in T$ and $T \cap K \neq \emptyset$. It is known that if S is a compact connected semigroup and $e \in E$, there exists a subsemigroup T of S such that $T \cap K \neq \emptyset$, T is an irreducible semigroup with identity e . Moreover T may be chosen to be in the centralizer of any abelian subgroup of H_e with $T \cap H_e = \{e\}$ [Hofmann and Mostert; 1966]. In this context it is convenient to say that T is an irreducible semigroup from e to K . The next theorem is a consequence of the existence of irreducible semigroups.

0.5 Theorem. Let S be a continuum satisfying $S = eS$ for some $e \in E$. If $e \in S \setminus K$, then e is peripheral.

Proof. Suppose U is an open set containing e disjoint from K . Select an irreducible semigroup T from e to K , and let $V = S \setminus T(S \setminus U)$. Then V is an open set containing e and is contained in U . Define the map $H : Tx(S, S \setminus V) \rightarrow (S, S \setminus V)$ by $H(t, x) = tx$. The conclusion follows from the generalized homotopy axiom [Wallace; 1953b].

Maximal Elements in Compact Semigroups. An element x in a semigroup S is \mathcal{L} -maximal, respectively, \mathcal{R} -maximal, \mathcal{J} -maximal, if it is maximal relative to the quasi-orderings $\leq (\mathcal{L})$, $\leq (\mathcal{R})$, or $\leq (\mathcal{J})$. If S is compact, then each element of S is below a maximal element; in particular maximal elements exists.

We now establish relations between maximal elements and maximal proper ideals. For this purpose it is convenient to introduce the following notation. For $A \subseteq S$ let $J_o(A) (L_o(A), R_o(A))$ denote the union of the family of all ideals (left, right ideals) contained in A . Thus, if $J_o(A) \neq \square$, it is the largest ideal contained in A . If S is compact, then $J_o(A)$ is open if A is open. Also if A is closed, $J_o(A)$ is closed; here S need not be compact.

0.6 Theorem. Let S be a semigroup and $a \in S \setminus K$; then (i) a is \mathcal{J} -maximal if and only if $J_o(S \setminus a)$ is a maximal proper ideal;
(ii) a is \mathcal{L} -maximal if and only if $L_o(S \setminus a)$ is a maximal proper left ideal;
(iii) if S is compact and a is \mathcal{J} -maximal, then a is \mathcal{L} -maximal.

Proof. Suppose $J_0(S \setminus a)$ is a maximal proper ideal, and $J(a) \subsetneq J(b)$ for some $b \in S$. Then $J_0(S \setminus a) \cup J(a) = S$, so $b \in J_0(S \setminus a) \cup J(a)$. Clearly, $b \in J_0(S \setminus a)$, since then a would be an element of $J_0(S \setminus a)$; and hence $b \in J(a)$. Therefore $J(a) = J(b)$, so a is \mathcal{J} -maximal.

Conversely, suppose that a is \mathcal{J} -maximal, M is an ideal of S such that $J_0(S \setminus a) \subsetneq M \subsetneq S$. If $x \in S \setminus J_0(S \setminus a)$, then $a \in J(x)$ since $J_0(S \setminus a) \cup J(x)$ contains $J_0(S \setminus a)$ properly. But a is \mathcal{J} -maximal so $J(a) = J(x)$; in particular $x \in J(a)$. Therefore $J_0(S \setminus a) \cup J(a) = S$. It follows then that either $M = S$ or $M = J_0(S \setminus a)$. This completes the proof of (i).

Suppose now that S is compact, and a is \mathcal{J} -maximal. If $L(a) \subsetneq L(b)$ for some $b \in S$, then $J(a) \subsetneq J(b)$; thus $J(a) = J(b)$. If $b \in aS \cup SaS$, then $Sa \subsetneq Sb \subsetneq Say$ for some $y \in S$; so by theorem 0.3 $Sa = Sb$. Now by the fact that $a \cup Sa \subsetneq b \cup Sb$, either $a = b$ or $a \in sb$. In the latter case, $a \in Sa$ so that $aS \subsetneq SaS$; thus $b \in SaS = SbS$. Hence $b \in ESE$ by theorem 0.4 in which case it follows that $b \in Sb$. Therefore $b \in Sa$, and it is now easily verified that a is \mathcal{L} -maximal.

0.7 Remark. It follows immediately from this theorem that maximal ideals exist in compact semigroups. If M is a maximal ideal in S , then $M = J_0(S \setminus x)$ for each $x \in S \setminus M$. It follows then that $J(x) = J(y)$ for each x and y in $S \setminus M$. Therefore all elements outside the maximal ideal M lie in the same \mathcal{J} -class. Also if $x \notin M$, it is easily seen that $S \setminus \mathcal{J}_x = J_0(S \setminus x)$. Therefore the elements outside a maximal ideal M form a \mathcal{J} -class. Analogous statements are true if one considers maximal left ideals, etc.

0.8 Theorem. If S is a compact semigroup satisfying $S^2 = S$, and a is an \mathcal{L} -maximal (\mathcal{R} -maximal) element of S , then $a \in ES$ ($a \in SE$).

Proof. Since $S^2 = S$, $a \in Sb$ for some $b \in S$, and as a is \mathcal{L} -maximal we have that $b \in Sa$. Therefore $a \in Sa$, and the conclusion follows from theorem 0.4.

It has been shown by Koch and Wallace [1954] that each proper ideal in a compact semigroup is contained in a maximal proper ideal. Hence by remark 0.6 there exist a \mathcal{J} -maximal element outside each proper ideal of S . Since each \mathcal{J} -maximal element is \mathcal{L} -maximal there exists an element of ES outside each proper ideal of S ; the following theorem due to Koch and Wallace follows easily.

0.9 Theorem. If S is compact satisfying $S^2 = S$, then $S = SES$. Hence the complement of each ideal in S contains an idempotent.

Now suppose that M is a maximal ideal in a compact semigroup. As noted in remark 0.6 $M = J_0(S \setminus x)$ for each $x \in S \setminus M$, and hence M is open. If S is also connected, then M is dense, therefore each maximal \mathcal{J} -class is a continuum S is nowhere dense by remark 0.6. It follows easily that M is connected if $S^2 = S$, so we have the following theorem due to Koch and Wallace [1954; 1958a].

0.10 Theorem; Let S be a continuum satisfying $S^2 = S$. Each proper ideal is contained in a maximal proper ideal; and each such is dense, open, and connected.

The following corollary is an easy consequence of the preceding discussion.

0.11 Corollary. If S is a continuum satisfying $S^2 = S$, then $S \setminus \mathcal{L}_x$ is connected for each maximal element x in S . Thus $S \setminus \{x\}$ is connected for each maximal element x .

0.12 Proposition. Let S be compact and e an \mathcal{L} -maximal idempotent. If \mathcal{L}_e is open in the set of all \mathcal{L} -maximal elements, then $e \in (Se)^0$.

Proof. By hypothesis there exists an open set U containing \mathcal{L}_e and no other \mathcal{L} -maximal elements. The left ideal $S(S \setminus U)$ is closed and contains all maximal \mathcal{L} -classes except \mathcal{L}_e . As each element in S is below an \mathcal{L} -maximal element, it follows that the complement of $S(S \setminus U)$ is contained in Se . Therefore $e \in (Se)^0$, and the proof is complete.

Now suppose S is a compact semigroup satisfying $S^2 = S$, e a \mathcal{G} -maximal idempotent of S , and $b \in \mathcal{G}e$. Then $SbS = SeS$, so $b = peq$ for some elements p and q in S . It may be verified at the reader's pleasure that the map $\rho : sEs \rightarrow bSb$ is a homeomorphism onto carrying \mathcal{K}_e onto \mathcal{K}_b . The next theorem then follows easily.

0.13 Theorem. If S is compact satisfying $S^2 = S$, a and b are \mathcal{G} -maximal elements with $a\mathcal{G}b$, then aSa is homeomorphic to bSb by a homeomorphism carrying \mathcal{K}_a onto \mathcal{K}_b .

Cohomology of Compact Semigroups. As stated previously, the cohomology theory used here is Alexander-Spanier-Wallace theory. We state two theorems used in the sequel [Cohen and Koch; 1965].

0.14 Theorem. Let S be a continuum with a zero and L a left ideal of S . If $e^2 = e \in S$, then $H^p(eL) = 0$ for all $p \geq 0$.

0.15 Theorem. Let S be a continuum with a zero satisfying $S = ESE$. If I is a closed ideal of S , then $H^1(AI) = 0$ for each closed subset A of E ; in particular $H^1(I) = 0$.

CHAPTER I

LOCAL SEPARATING PROPERTIES OF MAXIMAL ELEMENTS

In this chapter we investigate a particular aspect of the local structure of topological semigroups: namely, the local cutting properties of maximal elements in compact connected semigroups satisfying $S^2 = S$. In the first part of this chapter it is shown that a maximal idempotent can be neither a weak cutpoint nor a local separating point. Later we make use of these results to give sufficient conditions under which maximal \mathcal{J} -classes fail to be local separating sets. For further information concerning the topological properties of local separating points and weak cutpoints in general, the reader is referred to Whyburn [1942].

1.1 Definition. A point p in a continuum X is a weak cut point between a and b if a and b are points in X different from p , and any subcontinuum of X containing a and b also contains p . The point p is simply a weak cutpoint if there exist points a and b such that p is a weak cutpoint between a and b .

We shall show now that an \mathcal{J} -maximal idempotent in a compact connected semigroup satisfying $S^2 = S$ cannot be a weak cutpoint. We shall employ the following theorem due to Hunter [1961] and Koch [1957b].

1.2 Theorem. If S is a compact connected semigroup with a right identity e , and S is not a minimal ideal, then e is not a weak cutpoint of S .

Proof. Since e is peripheral (theorem 0.5), S is 1-semi-locally connected at e (proposition 0.2). Hence for any two points a and b in $S \setminus \{e\}$, there exists an open set V containing e but not a or b such that $S \setminus V$ is connected. The conclusion is obvious.

1.3 Theorem. If S is a continuum with a zero 0 , satisfying $S^2 = S$, and $e^2 = e \in S$ is an \mathcal{L} -maximal element of S , then e is not a weak cutpoint of S .

Proof. Suppose that e is a weak cutpoint of S between a and b for some pair of elements a and b of S . Since $S^2 = S$, there exist elements x and y in S such that $a \in Sx$ and $b \in Sy$. But S has a zero, so $Sx \cup Sy$ is a subcontinuum of S containing a and b ; hence $e \in Sx \cup Sy$. As e is an \mathcal{L} -maximal element of S , it may be assumed that $Sx = Se$. Now either e is also an element of Sy , in which case $Se = Sy$; or e is not an element of Sy . In the first case e would be a weak cutpoint of Se between a and b in contradiction to theorem 1.2. We shall obtain a similar contradiction for the second case by showing e would be a weak cutpoint between a and 0 . With this end in mind select any subcontinuum of Se containing a and 0 . Then $M \cup Sy$ is a subcontinuum of S containing e . As e is not contained in Sy , it must be that $e \in M$. Therefore e is a weak cutpoint of Se between e and 0 , and again we obtain a contradiction. Hence e is not a weak cutpoint of S , and the proof is complete.

We remark that the conclusion holds if e is a \mathcal{J} -maximal element, since by theorem 0.6 each \mathcal{J} -maximal element is \mathcal{L} -maximal. The

following lemma is stated for the convenience of the reader.

1.4 Lemma. Let X be a continuum and p a weak cutpoint of X . If A is a subcontinuum of X contained in $X \setminus \{p\}$, $\rho : X \rightarrow X/A$ the natural surjection, then $\rho(p)$ is a weak cutpoint of X/A .

1.5 Corollary. Let S be a continuum satisfying $S^2 = S$. If e is a \mathcal{J} -maximal idempotent of S and e is a weak cutpoint of S , then $S = K$.

Proof. If $S \neq K$, consider the Rees quotient modulo K with the natural surjection $\rho : S \rightarrow S/K$. S/K is a compact connected semigroup with a zero satisfying $(S/K)^2 = (S/K)$, and $\rho(e)$ is a \mathcal{J} -maximal idempotent of S/K . Therefore, $\rho(e)$ is not a weak cutpoint of S/K , and by lemma 1.4 e is not a weak cutpoint of S . The proof is complete.

Recall that a set C in a continuum X is a c-set if for each subcontinuum M with $M \cap C \neq \emptyset$, either $C \subseteq M$ or $M \subseteq C$.

1.6 Corollary. Let S be a continuum satisfying $S^2 = S$ and e a \mathcal{J} -maximal idempotent of S . If $S \neq K$, then e lies on no non-trivial c-set of S .

Proof. Suppose C is a c-set of S and $e \in C$. If $C \setminus \{e\} \neq \emptyset$ and $S \setminus C \neq \emptyset$, there exist points a and b in $S \setminus \{e\}$ with $a \in C$ and $b \notin C$. It is readily verified that e is a weak cutpoint of S between a and b , a contradiction. Therefore either $C = \{e\}$ or $C = S$, and the proof is complete.

A continuum X is said to be indecomposable if it is not the union

of two proper subcontinua of X . A composant of a point p in a continuum X is the union of all proper closed connected subsets of X containing p . It is known that an indecomposable metric continuum contains at least two composants, and each composant is a non-degenerate c -set [Kuratowski; 1948]. Consequently we have the following corollary due to Koch and Wallace [1958b].

1.7 Corollary. If S is a metric indecomposable continuum satisfying $S^2 = S$, then $S = K$.

The next theorem concerning the cutting properties of maximal idempotents is an extension of a theorem of Koch [1957b].

1.8 Theorem. Let S be a continuum with $S^2 = S$ and satisfying the following:

- (i) $S \neq K$,
- (ii) $cdS \leq 1$,
- (iii) $H^1(S) = 0$,

If e is a \mathcal{J} -maximal element of S , then e does not separate any subcontinuum of S .

Proof. Let M be a subcontinuum of S , and suppose $M \setminus \{e\} = P \cup Q$ with $P^* \cap Q^* = \{e\}$. Select arbitrary points $a \in P$ and $b \in Q$. Since $S \neq K$, e is not a weak cutpoint of S ; hence there exists a subcontinuum N of S containing a and b but not e . Consider the following part of the Mayer-Victoris sequence (using reduced groups in dimension zero):

$$\rightarrow H^0(M) \times H^0(N) \rightarrow H^0(M \cap N) \rightarrow H^1(M \cup N) \rightarrow .$$

Noting that $M \cap N$ is not connected, we see that $H^1(M \cup N)$ is not trivial. By using the fact that $cdS \leq 1$, it follows that

$H^1(S) \neq 0$, a contradiction. The proof is thus complete.

We note that the conclusion holds if $S + ESE$, $cdS \leq 1$, and S has a zero; as in this case (i) and (ii) are satisfied (theorem 0.15).

We now proceed to give sufficient conditions under which \mathbb{Z} -maximal elements fail to be local separating points. The terminology here is that of Whyburn [1942].

1.9 Definition. A point p in a continuum X is a local separating point of X if p separates the component containing p of some closed neighborhood of p .

It is required that p cut components of "closed" neighborhoods for convenience only. It is easily seen from theorem 0.1 that if p is a local separating point of X , then p separates the components containing p of all small neighborhoods of p .

Now if the continuum X is locally connected at p and p is a local separating point of X , it is noted that p is an intrinsic point of X . Indeed, there exist a closed connected neighborhood of p such that $N \setminus p$ is separated. Then for any open set V containing p and contained in N with $N \setminus V$ not connected, the natural induced homomorphism $H^0(N) \rightarrow H^0(N \setminus V)$ is not an isomorphism. Hence p is an intrinsic point of N , and therefore of X . J. D. Lawson has shown [unpublished] that X need not be locally connected for the same conclusion to hold. We thus have the following proposition.

1.10 Proposition. If X is a continuum (locally connected at p) and p is a local separating point of X , then p is an intrinsic point of X .

Now if T is an irreducible semigroup with identity e , then T is locally connected at e [Hoffman and Mostert; 1966]. As previously noted e is peripheral in T ; thus we have the following proposition.

1.11 Proposition. If T is an irreducible semigroup with identity e , then e is not a local separating point of e .

1.12 Definition. Let S be a semigroup. For $a \in S$, define P_a to be the set of left units for a ; i.e., $P_a = \{x \in S : xa = a\}$.

1.13 Remark. If S is compact and S^2 is S , then $P_a \neq \emptyset$ for each \mathcal{L} -maximal element a of S (theorem 0.8). Therefore in this case P_a is a compact semigroup and, consequently, has a minimal ideal, which we denote by K_a .

1.14 Lemma. Let S be a continuum with $S^2 = S$, and a an \mathcal{L} -maximal element of S not contained in K . If f is an idempotent of K_a and T is an irreducible semigroup from f to K , then

- (i) if $a = tx$ for some $t \in T$, $x \in S$, then $t = f$; thus if $x \in fS$, $x = a$.
- (ii) if $a \in xTa$ for $x \in S$, the $xTa \subseteq gSa$ for some $g \in K_a$.

Proof. T being irreducible from f to K implies that $T \cap \mathcal{J}_f = \{f\}$, so in particular $T \cap P_a = \{f\}$. Indeed, if $x \in SfS \cap P_a$, then $f \in P_a \times P_a \subseteq SxS$, in which event it follows that $J(x) = J(f)$. Therefore if $x \in T \cap P_a$, $x \in SfS \cap P_a$; so $x \in T \cap \mathcal{J}_f$, which implies that $T \cap P_a = \{f\}$. Suppose now that $a = tx$ for some $t \in T$, $x \in S$. Then $a \in Sta \subseteq Sx$, and by the maximality of a we have that $x \in Sa$.

Therefore $a = tga$ for some $g \in S$, so $tg \in P_a \cap SfS$. By the above observation $f \in StgS \subseteq StS$, thus $J(f) = J(t)$, so that $t = f$.

This proves part (i).

Now if $a \in xTa$ for some $x \in S$, then $xt \in Sff \cap P_a$ for some $t \in T$. As in the proof of (i) it follows that $t = f$, and so $xf \in P_a$. Since $f \in K_a$ we have that $xf \in K_a$, and the conclusion follows from the fact that K_a is the union of groups. This completes the proof of (ii).

The next theorem, although somewhat limited in its application, does have something to say concerning the local separating properties of maximal elements. We shall make use of it in a later example.

1.15 Theorem. Let S be a continuum satisfying $S^2 = S$. If a is an \mathcal{L} -maximal element of S not contained in K , and f is any idempotent contained in K_a , then a is peripheral if fS .

Proof. Let T be an irreducible semigroup from f to K , and suppose U is a relative open set of fS containing a while disjoint from K . Let $V = fS \setminus T(fS \setminus U)$, and note that V is a relative open set of fS containing a . Indeed if $a = tx$ for some $t \in T$, $x \in fS \setminus U$, then by lemma 1.14 $t = f$ and $x = a$, an obvious contradiction. Moreover V is contained in U as $fS \setminus U$ is contained in $T(fS \setminus U)$. Define $H : T \times (fS, fS \setminus V) \rightarrow (fS, fS \setminus V)$ by $H(t, x) = tx$, and the conclusion follows from the generalized homotopy axiom.

1.16 Lemma. Let S be a continuum satisfying $S^2 = S$, $a \in S \setminus K$ an \mathcal{L} -maximal element. If a is a local separating point of S , then a is a local separating point of Sa .

Proof. Suppose that N is a closed neighborhood of a such that $C \setminus \{a\} = P \cup Q$ with $P^* \cap Q^* = \{a\}$, where C is the component of N containing a . It may be assumed that $N \cap K = \emptyset$. Let M denote the component of $Sa \cap N$ containing a . To show that a is a local separating point of Sa it suffices to show that $M \cap P \neq \emptyset \neq M \cap Q$.

To this end select a net (x_α) in P converging to a . Since $S^2 = S$, there exists a net (y_α) (with same domain) in S such that $x_\alpha \in Sy_\alpha$ for each α . In view of the compactness of S it may be assumed that (y_α) converges to $y \in S$. Now denote the component of $Sy_\alpha \cap N$ containing x_α by M_α . If $a \in M_\alpha$ for any α , then by the maximality of a , $Sa = Sy_\alpha$; that is to imply $M = M_\alpha$. In this case it is obvious that $M \cap P \neq \emptyset$. Otherwise $a \notin M_\alpha$ for each α , and under this assumption it follows that $M_\alpha \subset P$ for each α . Since $N \cap K = \emptyset$, $N \cap Sy_\alpha$ is a proper subset of Sy_α ; and by theorem 0.1 we have that $M_\alpha \cap F(N) \neq \emptyset$. Combining these observations yields that $\limsup M_\alpha \subset \bar{P}$, and $\limsup M_\alpha \cap F(N) \neq \emptyset$. Because $a \in \liminf M_\alpha$, $\limsup M_\alpha$ is connected. As the net (y_α) converges to y , $\limsup Sy_\alpha = Sy$. But $a \in Sy$ and its being \mathcal{L} -maximal implies $Sy = Sa$.

Therefore $\limsup M_\alpha$ is a connected subset of Sa , contained in \bar{P} and meeting the boundary of N . It then follows that $\limsup M_\alpha \subset M$ so $M \cap P \neq \emptyset$. As the labeling of the separating sets P and Q was

arbitrary, we have that $M \cap P \neq \emptyset$ and $M \cap Q \neq \emptyset$. Therefore a is a local separating set of S_a , and the proof is complete.

1.17 Remark. As a consequence of the proof of this lemma, not only is the point a a local separating point of S_a , but the local separation of S_a is compatible with the local separation of S . That is, if N is a closed neighborhood of S in which a separates a component, C , then $N \cap S_a$ is a relative closed neighborhood of S_a in which a cuts its component M . Moreover, if $C \setminus \{a\} = P \cup Q$ with $P^* \cap Q^* = \{a\}$, then $M \setminus \{a\} = (P \cap M) \cup (Q \cap M)$ with $(P \cap M)^* \cap (Q \cap M)^* = \{a\}$.

1.18 Theorem. Let S be a continuum satisfying $S^2 = S$, a an \mathcal{L} -maximal element of S not contained in K . If a is a local separating point of S , then a is a local separating point of $gS \cup hS$ for some pair of idempotents g and h contained in K_a .

Proof. Suppose N is a closed neighborhood of a disjoint from K , and that a separates the component C of N containing a . Thus $C \setminus \{a\} = P \cup Q$ and $P^* \cap Q^* = \{a\}$. Now for each idempotent e of K_a , denote the component of $eS \cap N$ containing a by M_e . Because $N \cap (gS \cup hS)$ is a relative closed neighborhood of a in $gS \cup hS$ for each pair of idempotents g and h in K_a , it suffices to show that $M_g \cap P \neq \emptyset$ and $M_h \cap Q \neq \emptyset$ for some idempotents g and h in K_a .

With this end in mind select an idempotent f in K_a and an irreducible semigroup from f to K . By the preceding remark (1.17) there exists a net (x_α) in $S_a \cap P$ converging to a . Since $STa = S_a$, there exists

a net (y_α) in S (with same domain) such that $x_\alpha \in y_\alpha Ta$ for each α . As S is compact it may be assumed that (y_α) converges to some element y in S . Let M_α denote the component of $y_\alpha Ta \cap N$ containing x_α . If $a \in M_\alpha$ for some α , then by lemma 1.14 $y_\alpha Ta \subseteq gS$ for some idempotent g in K_a . Therefore, in this case, $M_\alpha \subseteq M_g$ so $M_g \cap P \neq \emptyset$. Otherwise, it may be assumed that $a \notin M_\alpha$ for each α . In this case $M_\alpha \subseteq P$ for each α , and so $\limsup M_\alpha \subseteq \bar{P}$. It is readily verified now that $\limsup M_\alpha \subseteq yTa \cap \bar{P}$ and that $\limsup M_\alpha \cap F(N) \neq \emptyset$. Also, since $a \in \liminf M_\alpha$, $\limsup M_\alpha$ is connected. Using lemma 1.14 once again, we have that $yTa \subseteq gS$ for some idempotent g in K_a ; thus $\limsup M_\alpha \subseteq M_g$, so it follows that $M_g \cap P \neq \emptyset$. Similarly it is seen that there exists an idempotent h in K_a such that $Q \cap M_h \neq \emptyset$, and the proof is complete.

Now assume that K_a is right simple; i.e., K_a is a minimal right ideal of itself. Then $fg = g$ and $gf = f$ for each pair of idempotents g and f in K_a . Thus $gS = fgS \subseteq fS = gfS \subseteq gS$, which implies that $gS = fS$ for each pair of idempotents in K_a . Combining this observation with theorem 1.15 and proposition 1.10, we obtain the following corollary.

1.19 Corollary. Let S be a locally connected continuum satisfying $S^2 = S$, $a \in S \setminus K$ an \mathcal{L} -maximal element of S . If K_a is right simple, then a is not a local separating point of S .

Proof. If a is a local separating point of S , a is a local separating point of $gS \cup hS$ for some pair of idempotents g and h in K_a . Since K_a is right simple, $gS \cup hS = fS$ for any idempotent

f in K_a ; so a is a local separating point of fS for each idempotent $f \in K_a$. But fS is a locally connected continuum, and thus a is an intrinsic point of fS by proposition 1.10. This is in contradiction to theorem 1.15; therefore a must not be a local separating point of S .

This last corollary may be obtained from theorem 5 of [Cohen and Koch; 1965]. However we were led to it here in a natural manner by different methods. We proceed now to one of the main theorems of this chapter.

1.20 Theorem. Let S be a continuum with $S^2 = S$, a an \mathcal{L} -maximal element of S , $a \in SE$. If a is a local separating point of S , then $a \in K$.

Proof. If $a \in SE$, by the maximality of a it follows that $Sa = Se$ for some $e \in E$. If a were a local separating point of S and not contained in K , then a would be a local separating point of Sa (lemma 1.16). Select an irreducible semigroup T from e to K . A contradiction is established by showing e would be a local separating point of T . As the map $t \rightarrow at$ is a homeomorphism of T onto aT carrying a to e , it suffices to show a is a local separating point of aT .

With this end in mind choose a relative closed neighborhood N of a in Sa , $N \cap K = \emptyset$, and suppose a separates the component C of N containing a ; i.e., $C \setminus \{a\} = P \cup Q$ with $P^* \cap Q^* = \{a\}$. Denote the component of $aT \cap N$ containing a by M . We intend to show that $M \cap P \neq \emptyset \neq M \cap Q$. Select a net (x_α) in P converging to a ; denote

the component of $x_\alpha T \cap N$ by M_α . If $a \in M_\alpha$ for any α , then $a = x_\alpha t$ for some $t \in T$; i.e., $a \in St$. Since a is \mathcal{L} -maximal, this would imply that $Sa = St$, in which case $J(a) = J(t)$. But because $T \cap \mathcal{J}_e = \{e\}$ this implies $t = e$, and so $x_\alpha = a$, a contradiction. Thus it may be assumed that $a \notin M_\alpha$ for each α . Using techniques similar to lemma 1.16, we obtain that $\limsup M_\alpha \subseteq aT \cap P^*$, $\limsup M_\alpha$ is connected, and $\limsup M_\alpha \cap F(N) \neq \emptyset$. Since $a \in \limsup M_\alpha$, this implies that $\limsup M_\alpha \subseteq M$, and hence $M \cap P \neq \emptyset$. Similarly it follows that $M \cap Q \neq \emptyset$, and so a is a local separating point of aT . All of this is a contradiction to the assumption that $a \notin K$, and the proof is complete.

1.21 Corollary. Let S be a continuum with a zero and satisfying $S = SE$. Then no \mathcal{L} -maximal element of S is a local separating point of S .

Proof. Since S has a zero and no \mathcal{L} -maximal elements equals zero, the conclusion follows immediately from the theorem.

The following gives the necessity of the hypothesis " a an \mathcal{L} -maximal element of S not contained in K " in the preceding theorems.

1.22 Example. The example is of a semigroup S satisfying $S = E$ and having an \mathcal{L} -maximal a contained in K which separates S .

Let I_0 denote the unit interval with right trivial multiplication (i.e., $xy = y$), and I_1 denote the unit interval with "min" multiplication (i.e., $xy = \min\{x, y\}$). Then $I_0 \times I_1$ is a semigroup under coordinatewise multiplication. Consider the subsemigroup S of $I_0 \times I_1$, $S = \{(x, y) \in I_0 \times I_1 : y \leq -2x + 1\} \cup \{(x, y) \in I_0 \times I_1 : y \leq 2 \times 1\}$ (see fig. 1). For each $(x_0, y_0) \in N$, $L(x_0, y_0) =$

$\{(x_0, y) \in N : y \leq y_0\}$ and $K = \{(x, y) \in N : y = 0, 0 \leq x \leq 1\}$. It is readily verified that the point $a = (\frac{1}{2}, 0)$ is an \mathcal{L} -maximal element of S , and a is a cutpoint of S .

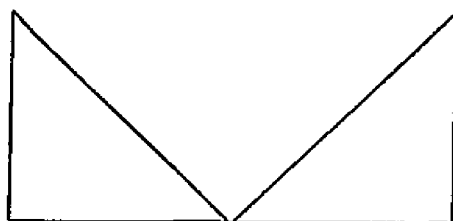


Fig. 1.

If our attention is shifted from \mathcal{L} -maximal elements to \mathcal{J} -maximal elements, we obtain the next result as a corollary of theorem 1.20.

1.23 Corollary. Let S be a continuum satisfying $S^2 = S$ and $S \neq K$. If a is a \mathcal{J} -maximal element of S , then a is not a local separating point of S .

Proof. If a is \mathcal{J} -maximal and $S \neq K$, then $a \notin K$. Since a is \mathcal{J} -maximal, by theorem 0.6 a is \mathcal{K} -maximal. Thus by theorem 0.8 $a \in SE$. But also by theorem 0.6 a is \mathcal{L} -maximal; so by combining this with theorem 1.20, it follows that a is not a local separating point of S .

As a consequence of this corollary, if S is a continuum such that every point of S is a local separating point of S , then $S^2 = S$ implies $S = K$. Some examples covered by this are the following: the 1-sphere S^1 ; three tangent S^1 's; S^1 together with a diameter; two disjoint S^1 's joined by an arc; and variations of these in which arcs are replaced by "long" arcs. (Some of these were known to Koch and Wallace [1958]).

Before proceeding to the local separating properties of maximal \mathcal{J} -classes, we give an application of some of the preceding theorems.

1.24 Example. Let S be the space of tangent circles converging to a point e , as illustrated in fig. 2.



Fig. 2.

S may be described as the one point compactification of $U \{S_n : n > 0\}$, where S_n is the circle centered at the point $(n,0)$ of the plane having radius $\frac{1}{2}$; the point at infinity is denoted e . The claim is that if S admits the structure of a semigroup satisfying $S^2 = S$, then $S = K$.

Assume now that $S \neq K$, and through a series of observations we shall arrive at a contradiction. As no maximal element is a local separating point of S (corollary 1.23), e must be the unique maximal element of S . But $S \setminus \{e\}$ is then the unique maximal proper ideal of S (remark 0.7); and because the complement of each ideal contains an idempotent (theorem 0.9), $e^2 = e$. Since each proper ideal is contained in a maximal proper ideal, and because $S \setminus \{e\}$ is the unique maximal proper ideal, it must be that $S = SeS$.

We thus have a semigroup S with a maximal idempotent e such that $S = SeS$. Another multiplication, \circ , is introduced on S as follows: for x and y elements of S , define $xoy = xey$. Then (S, \circ) is easily

seen to be a topological semigroup with the multiplication \circ satisfying $xoy = xoeoy$; i.e., e is a "middle" unit of (S, \circ) . Also $S \setminus \{e\}$ is an ideal in (S, \circ) ; in particular the minimal ideal K' of (S, \circ) is not all of S . Thus (S, \circ) is a topological semigroup satisfying the same general properties as the original semigroup (namely $S \neq K'$ and $SoS = S$) with the additional property that e is a middle unit. Thus we may assume that the semigroup S satisfies $S = SeS$, $S \neq K$ and $xy = xey$ for each $x, y \in S$.

Suppose now that f is an idempotent in S ; then $f = ff = fef$. Therefore the maps $\rho: fS \rightarrow efS$ and $\lambda: efS \rightarrow fy$ are homeomorphisms. Indeed for $x \in fS$, $\lambda\rho(x) = fex = fx = x$; and for $y \in efS$, $y = efq$ for some $q \in S$; so $\rho\lambda(y) = \rho\lambda(efq) = efefq = efq = y$. Since e is clearly an intrinsic of S , it follows by theorem 0.5 (and the fact that peripherality is a local property) that e is not an element of $(Se)^{\circ}$. Thus by proposition 0.12 it follows that e is not an isolated point of the \mathcal{L} -maximal elements of S . Combining theorem 0.10 with remark 0.7, we have that the complement of each maximal \mathcal{L} -class is open, dense, and connected. Hence each maximal \mathcal{L} -class is closed and meets each S_n in at most one point. Consequently each maximal \mathcal{L} -class contains only a finite number of elements. As e is not an isolated point of the \mathcal{L} -maximal elements, each neighborhood of e contains all but finitely many of the maximal \mathcal{L} -classes.

Let V be a very small neighborhood of e and p an \mathcal{L} -maximal element of S contained in V . Because $p \in Sp$ (theorem 0.8) and $Sp = Sep$, it is immediate that $\mathcal{L}_p = \mathcal{L}_{ep}$. By the preceding discussion we may assume that $\mathcal{L}_p \subset V$; hence we may assume $p \in eS$.

By theorem 1.18 p is a local separating point of $gS \cup fS$ for some pair of idempotents g and f in K_p . Since (i) $gep = gp = p$, $fep = fp = p$, and (ii) $gS = geS$, $fS = feS$; it follows that ge and fe are in K_p , and p is a local separating point of $geS \cup feS$. We assume then that $ge = g$ and $fe = f$. Note that p is peripheral in gS and fS (theorem 1.15), so p is not a local separating point of either gS or fS (proposition 1.10). A picture would now be appropriate.

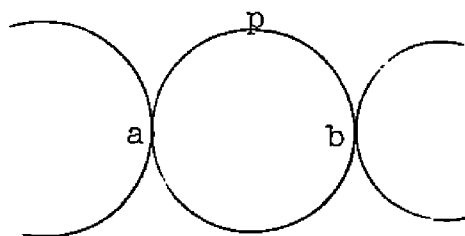


Fig. 3.

As p is very near e , K is disjoint from S_n as indicated in fig. 3. Let A denote the arc from a to b not containing p , B the arc from p to a containing b , and C the arc from p to a not containing b . Since p is a local separating point of $gS \cup fS$ and not of gS or fS , and whereas $S_n \cap K = \emptyset$, the arc B is contained in either fS or gS . Assume $B \subset fS$, in which case C must be contained in gS . Finally because eSe is connected, containing e and meeting K , the arc A is contained in eSe .

Now the map $\rho_f: fs \rightarrow efS$, defined by $\rho_f(x) = ex$, is a homeomorphism onto leaving the points p , a , and b fixed. Therefore $\rho_f(B)$ is an arc with end points a and p , containing the point b ; and so $B = \rho_f(B) \subset eS$. Also the map $\rho_g: gS \rightarrow egS$ defined by $\rho_g(x) = ex$

is a homeomorphism onto leaving p and b fixed. Thus $\rho_g(C)$ is an arc with end points p and b . By theorem 0.14 it is readily verified that S_n is not contained in eS since $S_n \cap K = \emptyset$. But $\rho_h(C) \subseteq eS$ and is an arc with endpoints a and p ; therefore $\rho_g(C) = B$. That is to say $eC = B$. Let $x_0 \in C$ such that $ex_0 = b$, in which case $gb = gex_0 = gx_0 = x_0$.

With the structure of S being thus determined, select a thread T from e to K [Koch; 1964]. Because p was chosen near e , b must be an element of T . Thus $x_0 = gb$ is an element of gT which is an arc with one end point ge , the other endpoint an element of K . Since $p \notin Se$ (theorem 1.20), it is obvious that $p \in gT$, and hence the endpoint ge must be contained in the arc with end points x_0 and p and not containing b . It is clear then that ge is a local separating point of gS . But $ge = g$ is a \mathcal{J} -maximal element of the semigroup gS . This establishes a contradiction to corollary 1.23, and therefore the assumption that $S \neq K$ must be false. Q.E.D.

A few things of interest might be pointed out concerning techniques of the previous example. First it should be noted that if a space admits the structure of a semigroup S with $S = SeS$ for some idempotent e , then it admits the structure of a semigroup such that e is a "middle" unit. Moreover if e is a middle unit in S , then the map $x \rightarrow ex$ is a homomorphism of S onto eS and this map restricted to fS is a homeomorphism for each idempotent f in S . Thus the study of semigroups with middle units may possibly be a useful technique in the study of semigroups S with $S^2 = S$. For the last part of this chapter we shall concern ourselves with the local separating properties of maximal \mathcal{J} -classes. Our interest

in this is to give conditions under which a maximal idempotent cannot have a 2-dimensional Euclidean neighborhood.

1.25 Definition. A set A in a continuum X is a local separating set at the point p if there exists small open sets V of p such that A separates the component of V containing p .

1.26 Theorem. Let S be a continuum, $S \neq K$, and satisfying $S = SeS$ for some idempotent e . If \mathcal{K}_e is connected and abelian, then \mathcal{K}_e is not a local separating set of S at any point of \mathcal{K}_e .

Proof. Since \mathcal{K}_e is connected and abelian, there exists an irreducible semigroup T from e to K such that $hT = Th$ for each $h \in \mathcal{K}_e$ [Hofmann and Mostert; 1966, theorem IV]. Suppose $a \in \mathcal{K}_e$, V is an open set containing a and disjoint from K . Let C denote the component of V containing a and suppose further that $C \setminus \mathcal{K}_e = P \cup Q$ with $P^* \cap Q = \emptyset = P \cap Q^*$. We shall first show that $C \cap P^* \cap Q^* \neq \emptyset$. With this end in mind suppose $C \cap P^* \cap Q^* = \emptyset$. Then $C \cap P^*$ and $C \cap Q^*$ are relatively closed subset of C and as they are disjoint their union is not C . Thus there exists a point b in $C \setminus (P^* \cup Q^*)$. Let U be an open set containing b such that $U^* \subseteq V$ and $U^* \cap (P^* \cup Q^*) = \emptyset$. Let A be a connected subset of U^* such that $A \cap \mathcal{K}_e = \{b\}$ and $A \cap F(U) \neq \emptyset$. (A may be taken to be the component of $bT \cap U^*$ containing b). Then $A \subseteq V$ and hence $A \subseteq C$. Since $A \cap (P^* \cap Q^*) = \emptyset$, A must be contained in \mathcal{K}_e , a contradiction. Therefore there exists a point b in $C \cap P^* \cap Q^*$.

We shall now obtain a contradiction to proposition 1.11 by showing e is a local separating point of T under these conditions. As the

map $t \rightarrow tb$ is a homeomorphism of T onto Tb taking e to b , it suffices to show that b would be a local separating point of Tb . With this end now in mind let N be a closed neighborhood of b contained in V , and denote the component of $Tb \cap N$ containing b by M . All that needs to be shown is that $M \cap P \neq \emptyset \neq M \cap Q$, since $M \setminus \{e\} = M \setminus \mathcal{N}_e \subseteq P \cup Q$.

Select a net (x_α) in $P \cap N$ converging to b . Since $S = SeS$, there exists nets (p_α) and (q_α) in S such that $x_\alpha = p_\alpha e q_\alpha$. In view of the compactness of S , it may be assumed that (p_α) converges to p and that (q_α) converges to q for some p and q in N . Clearly $b = peq$, and by the maximality of b it follows easily that $pe \in eS$ and $eq \in Se$; in particular $pe \in \mathcal{N}_e$ and $eq \in \mathcal{N}_e$. Therefore $pTq = peTeq = Tpeq = Tb$ and so $\limsup p_\alpha Tq_\alpha = \liminf p_\alpha Tq_\alpha = Tb$. Let M_α denote the component of $p_\alpha Tq_\alpha \cap N$ containing x_α . If $M_\alpha \cap \mathcal{N}_e \neq \emptyset$, then since $T \cap \mathcal{J}_e = \{e\}$ and $S \setminus \mathcal{J}_e$ is an ideal it would follow that $p_\alpha e \in \mathcal{N}_e$ and $q_\alpha e \in \mathcal{N}_e$. This of course would imply that $x_\alpha \in \mathcal{N}_e$, a contradiction. Therefore $M_\alpha \cap \mathcal{N}_e = \emptyset$, and so $M_\alpha \subseteq P$. By theorem 0.1 $M_\alpha \cap F(N) \neq \emptyset$. It follows then that $\limsup M_\alpha \subseteq P^* \cap Tb \cap N$ and $\limsup M_\alpha \cap F(N) \neq \emptyset$. As $b \in \liminf M_\alpha$, $\limsup M_\alpha$ is connected and is therefore contained in M . Therefore $M \cap P \neq \emptyset$; also it may be shown that $Q \cap M \neq \emptyset$, so b is a local separating point of Tb . This completes the proof.

1.27 Lemma. If S is a continuum with identity e , and S is not a group, then \mathcal{N}_e is not a local separating set of S at any point of \mathcal{N}_e .

Proof. Everything being the same as the proof of theorem 1.26, let $p_\alpha = e$ for each α . It is then noted that the condition " $hT = Th$ for each $h \in \mathcal{K}_e$ " is not needed, and we finish the proof in the same manner.

1.28 Corollary. Let S be a continuum satisfying $S^2 = S$ and $S \neq K$, if e is a \mathcal{J} -maximal idempotent of S , $a \in \mathcal{J}_e$, then \mathcal{K}_a is not a local separating set of aSa at any point of \mathcal{K}_a .

Proof. Since $a \in \mathcal{J}_e$, it follows from theorem 0.13 that aSa is homeomorphic to eSe by a homeomorphism taking \mathcal{K}_a onto \mathcal{K}_e . The conclusion follows from lemma 1.27.

1.29 Theorem. Let S be a continuum satisfying $S^2 = S$ and $S \neq K$. If e is a maximal idempotent of S with \mathcal{K}_e totally disconnected, then \mathcal{J}_e is not a local separating set of S at any point of \mathcal{J}_e .

Proof. Let $a \in \mathcal{J}_e$ and V an open set of disjoint from K . Suppose that $C \setminus \mathcal{J}_e = P \cup Q$ with $P^* \cap Q = \emptyset = P \cap Q^*$. As in the proof of theorem 1.26, there exist an element b in $C \cap P^* \cap Q^*$. Let U be an open set of b such that $\bar{U} \subseteq V$, and let M denote the component of $U \cap bSb$ containing b . We shall obtain a contradiction by showing \mathcal{K}_b is a local separating set of bSb at b for which it suffices to show $M \cap P \neq \emptyset \neq M \cap Q$. With this end in mind select a net (x_α) in $P \cap U$ converging to b . Then there exist nets (p_α) and (q_α) in S such that $x_\alpha \in p_\alpha S q_\alpha$. Because S is compact, it may be assumed that (p_α) converges to p and (q_α) converges to q for some p and q in S . Then $p_\alpha S q_\alpha$ converges to pSq and $b \notin pSq$. Since b is \mathcal{J} -maximal, it follows

that $pSq = bSb$. Let M_α denote the component of $U \cap p_\alpha Sq_\alpha$ containing x_α . If $b \in M_\alpha$ for any α , then $p_\alpha Sq_\alpha = bSb$, so $M = M_\alpha$; and we then have $M \cap P \neq \emptyset$. This is what was to be shown. Now it is assumed that $b \notin M_\alpha$ for each α , in which case $M_\alpha \subset P$ for each α . As before we obtain that $\limsup M_\alpha$ is a non-degenerate continuum contained in $P^* \cap bSb \cap \bar{U}$. Since \mathcal{K}_b is totally disconnected, $\limsup M_\alpha$ is not contained in \mathcal{K}_b ; it may be verified that $\limsup M_\alpha \cap P \neq \emptyset$, from which it follows $M \cap P \neq \emptyset$. In a similar manner it is shown that $M \cap Q \neq \emptyset$, and so \mathcal{K}_b is a local separating set of bSb . This establishes the contradiction, and the proof is complete.

CHAPTER 2

SEMIGROUPS ON FINITELY FLOORED SPACES

Cohen and Koch [1965] have shown that if S is a topological semigroup on a compact connected two-manifold without boundary satisfying $S = ESE$, then $S = K$. In this chapter we generalize this to a class of finitely floored spaces. We proceed with a theorem of Wallace [1961] stated in a slightly different version of its original form.

If A is a closed subset of a space X and h is an element of $H^n(X)$, then $h|_A$ denotes the image of h under the natural homomorphism $H^n(X) \xrightarrow{i^*} H^n(A)$.

2.1 Theorem. Let X and Y be compact spaces, and \leq a closed relation from X to Y such that $L(A) \cap L(B)$ is connected for each pair of closed subset A and B of X . If $h \in H^1(X)$ has the property that $h|_{L(y)} = 0$ for each $y \in Y$, then $h|_{L(A)} = 0$ for each closed subset A of Y .

Proof. Suppose that the conclusion is false. Select a subset F of Y minimal relative to being closed and satisfying $h|_{L(F)} \neq 0$. Since $h|_{L(y)} = 0$ for each $y \in Y$, F is not a single element; therefore $F = A \cup B$ where A and B are proper closed subsets of F . Consider the exact sequence:

$$\rightarrow H^0(L(A) \cap L(B)) \xrightarrow{\Delta} H^1(L(F)) \xrightarrow{J^*} H^1(L(A)) \times H^1(L(B)) \rightarrow \dots$$

By the minimal conditions on F , $h|_{L(A)} = 0$ and $h|_{L(B)} = 0$; thus $J^*(h|_{L(F)}) = 0$. But $L(A) \cap L(B)$ is connected, so J^* is 1-1, and

therefore $h|_L(F) = 0$. This is a contradiction to the hypothesis by which F was chosen; hence $h|_L(A) = 0$ for each closed subset A of Y .

2.2 Definition. Let X be a space and L a subset of $H^n(X)$. A closed subset F of X is a floor for L if (i) $h|_F = 0$ for each $h \in L$ and (ii) for each proper closed subset A of F , there exists an element h in L such that $h|_A = 0$. If L consists of a single element h , F is called a floor for h .

1.1. If X is a compact space and $L \subseteq H^n(X)$ not containing 0, then there exists a floor F for L . In particular if A is a closed subset of X such that $h|_A \neq 0$ for each $h \in L$, then F may be chosen as a subset of A . By using this, it is easily verified that a floor for L is the closure of a union of floors for elements of L .

Indeed suppose that F for L . For each $h \in L$ select a floor F_h for h which is contained in F , and let $A = (\cup \{F_h : h \in L\})^*$. Then it is easily seen that $h|_A \neq 0$ for each $h \in L$; and since A is a closed subset of F , it follows that $A = F$.

The following topological lemma is a modified version of that due to Cohen and Koch [1965].

If A is a closed subset of a space X , recall that X/A denotes the space obtained from X by shrinking A to a point.

2.3 Lemma. Let X be a continuum such that $H^n(X) \neq 0$, and suppose X is a floor for some subset L of $H^n(X)$. If A is a proper retract of X , then $H^n(X/A) \neq 0$ and X/A is a floor for some subset L_0 of $H^n(X/A)$. Moreover L_0 may be chosen to be finite if L is finite.

Proof. Let $\varphi: X \rightarrow X/A$ denote the natural map and $p = \varphi(A) \in X/A$. Consider the commutative diagram in which the top row is exact and φ_0^* is induced by the natural map $\varphi_0: (X, A) \rightarrow (X/A, p)$.

$$\begin{array}{ccccc} H^n(X, A) & \rightarrow & H^n(X) & \rightarrow & H^n(A) \\ \varphi_0^* \uparrow & & \varphi^* \uparrow & & \\ H^n(X/A, p) & \rightarrow & H^n(X/A) & & \end{array}$$

Note that by the map excision theorem [Wallace; 1952], φ_0^* is an isomorphism. Let $L_1 \subseteq L$ be those elements h in L satisfying $h|_A = 0$. For each $h \in L_1$ there exists an element h_0 in $H^n(X/A)$ such that $\varphi^*(h_0) = h$; thus $H^n(X/A) \neq 0$. Let $L_0 \subseteq H^n(X/A)$ be a "section" of L_1 ; i.e., for each $h \in L_1$ there exists a unique element h_0 in L_0 such that $\varphi^*(h_0) = h$. Clearly such a set exists and is finite if L is finite. We shall show that X/A is a floor for L_0 .

By the manner in which L_0 was chosen it is clear that L_0 does not contain 0. Suppose now that B is a proper closed subset of X/A and consider the commutative diagram

$$\begin{array}{ccc} H^n(X) & \rightarrow & H^n(\varphi^{-1}(B)) \\ \varphi^* \uparrow & & \varphi_1^* \uparrow \\ H^n(X/A) & \rightarrow & H^n(B) \end{array}$$

where φ_1^* is induced by the restriction of φ to $\varphi^{-1}(B)$. Since X is connected, $\varphi^{-1}(B) \cup A$ is a proper closed subset of X . Hence there exists an element h in L such that $h|_{\varphi^{-1}(B) \cup A} = 0$; in particular $h|_{\varphi^{-1}(B)} = 0$ and $h|_A = 0$. Because $h|_A = 0$, h is an element of L_1 so there exists an element h_0 in L_0 such that $\varphi^*(h_0) = h$. Our intention is to show that $h_0|_B = 0$. By the fact that

$h|_{\varphi^{-1}(B)} = 0$ and that $h|_{\varphi^{-1}(B)} = \varphi^*(h_0)|_{\varphi^{-1}(B)} = \varphi_1^*(h_0|_B)$, it suffices to show that φ_1^* is 1-1.

With this end in mind, two cases are considered: (i) $p \neq \varphi(A) \in B$ and (ii) $p = \varphi(A) \in B$. In case (i) φ_1 is a homeomorphism of $\varphi^{-1}(B)$ onto B and thus φ_1^* is an isomorphism. In case (ii) we consider the commutative diagram,

$$\begin{array}{ccc} H^n(B) & \xrightarrow{\varphi_1^*} & H^n(\varphi^{-1}(B)) \\ k^* \uparrow & & \uparrow i_0^* \\ H^n(B, p) & \xrightarrow{\varphi_2^*} & H^n(\varphi^{-1}(B), A) \end{array}$$

where φ_2^* is induced by the restriction of φ_0 to $(\varphi^{-1}(B), A)$.

Now k^* is an isomorphism and φ_2^* is an isomorphism by the map excision theorem [Wallace; 1952]. We claim that i_0^* is 1-1; consider the exact sequence:

$$\rightarrow H^{n-1}(\varphi^{-1}(B)) \xrightarrow{m^*} H^{n-1}(A) \rightarrow H^n(\varphi^{-1}(B), A) \xrightarrow{i_0^*} H^n(\varphi^{-1}(B)) \rightarrow .$$

Since A is a retract of X , m^* is onto. Therefore i_0^* is 1-1, and it follows that φ_1^* is 1-1. Therefore $h_1|_B = 0$, and the proof is complete.

The following theorem is of use in the sequel and is of interest in itself.

2.4 Theorem. Let S be a continuum with a zero satisfying $S = ESE$. If h is a non-zero element of $H^2(S)$, then there exist a pair of idempotents e and f in S such that $h|_{Se \cup Sf} \neq 0$.

Proof. Let F be a subset of E minimal relative to (i) being closed and (ii) satisfying $h|_{SF} \neq 0$. (Here we are using the hypothesis that $S = SE$). Since S has a zero, $H^2(Se) = 0$, so F does not consist of a single point. Thus F may be expressed as the union of two proper closed subsets A and B of F . Consider the Mayer-Victoris sequence,

$$\rightarrow H^1(SA \cap SB) \xrightarrow{\Delta} H^2(SF) \xrightarrow{J^*} H^2(SA) \times H^2(SB) \rightarrow .$$

Because of the minimal conditions on F , $h|_{SA} = 0$ and $h|_{SB} = 0$; thus $J^*(h|_{SF}) = 0$. Hence there exists an element h_0 in $H^1(SA \cap SB)$ such that $\Delta(h_0) = h$. Define a relation \leq from $SA \cap SB$ from $SA \cap SB$ to $A \times B$ as follows: for $x \in SA \cap SB$ and $(e, f) \in A \times B$ let $x \leq (e, f)$ if and only if $x \in Se \cap Sf$. It is easily verified that \leq is a closed relation from $SA \cap SB$ to $A \times B$. For each subset C of $A \times B$, $L(C) = \bigcup \{Se \cap Sf : (e, f) \in C\}$: therefore $L(C)$ is a left ideal in S . Because S has a zero and $S = ES$, it is easily verified that the left ideal $L(M) \cap L(N)$ is connected for each pair of closed subsets M and N of $A \times B$. Now $h_0|_{L(A \times B)} = h_0|_{SA \cap SB} = h_0 \neq 0$, so by theorem 2.1 it follows that there exists an element (e, f) in $A \times B$ such that $h_0|_{L((e, f))} \neq 0$; i.e., $h_0|_{Se \cap Sf} \neq 0$. Consider the commutative diagram:

$$\begin{array}{ccc} H^1(SA \cap SB) & \xrightarrow{\Delta} & H^2(SF) \\ H^1(Se) \times H^1(Sf) \rightarrow H^1(Se \cap Sf) & \xrightarrow{\Delta_0} & H^2(Se \cup Sf). \end{array}$$

Since S has a zero, Δ_0 is 1-1. Therefore $h|_{Se \cup Sf} = \Delta_0(h_0|_{Se \cap Sf}) \neq 0$, and the proof is complete.

Recall that the point p in a space X is peripheral if there exist small neighborhood V containing p such that $H^n(V^*, F(V)) = 0$ for all non-negative n . From proposition 0.2 and theorem 0.5 we have the following lemma.

2.5 Lemma. Let S be a continuum with e an idempotent of S/K . If $e \in (Se)^0$, then e is peripheral in S .

2.6 Lemma. Suppose that X is a continuum and that X is a floor for some subset L of $H^n(X)$; then each point of X is intrinsic.

Proof. For each non-empty subset V of X the natural homomorphism $H^n(X) \xrightarrow{i^*} H^n(X \setminus V)$ is not 1-1. Indeed since X is a floor for L , there exist an element h in L such that $h|_{X \setminus V} = 0$.

2.7 Theorem. Let S be a continuum satisfying $S = ESE$. If S is a floor for some finite subset of $H^2(S)$, then $S = K$.

Proof. Because S is a floor for some finite subset of $H^2(S)$, we have in particular that $H^2(S) \neq 0$. If $S \neq K$, then K is a proper retract of S [Wallace; 1957]; so the hypotheses of lemma 2.3 are satisfied. Thus S/K , the Rees quotient modulo K , satisfies the hypotheses; so it may be assumed that S has a zero.

Let L be a finite subset of $H^2(S)$ such that S is a floor for L . By theorem 2.4 for each $h \in L$ there exist idempotents e_h and f_h in S such that $h|_{Se_h \cup Sf_h} \neq 0$.

Let $A = \bigcup \{Se_h \cup Sf_h : h \in L\}$, and observe that $h|_A \neq 0$ for each $h \in L$. Because L is finite, A is closed; therefore $A = S$ since S

is a floor for L . We conclude the existence of a finite subset $\{e_1, e_2, \dots, e_n\}$ of E such that $S = \bigcup_{i=1}^n Se_i$. Clearly it may be assumed that $e_1 \notin Se_j$ for $1 \neq j$. Hence $e_1 \in (Se_1)^0$, so by lemma 2.5 e_1 is peripheral in S . This establishes a contradiction to lemma 2.6, and the proof is complete.

The class of compact connected 2-manifolds without boundary are covered by this theorem, and other examples are readily constructed. There is an example of a semigroup S with a zero satisfying $S = ESE$ and $H^2(S) \neq 0$ [Hudson; 1963]. The underlying space of S is a two-sphere with four closed intervals issuing from a common point, z , on the two-sphere. The point z is a zero for S and the other idempotents of S are the free endpoints of the four intervals. In view of theorems 2.7 and 2.4 this example is in a sense a prototype of any such example.

CHAPTER 3

A FURTHER RESULT ON MAXIMAL IDEMPOTENTS

It was shown by Mostert and Shields [1959] that an identity element of a compact connected semigroup S cannot have a Euclidean neighborhood unless S is a group. Later it was shown [Cohen and Koch; 1965] that a right identity of S cannot have a Euclidean neighborhood unless S is left simple. In this chapter, making use of results of chapters 1 and 2, we give conditions on S under which a maximal idempotent cannot have a two dimensional Euclidean neighborhood.

We proceed with the following topological lemmas.

3.1 Lemma. Let X be a space and suppose $X = A \cup B$ with $A \cap B = \{p\}$ for some $p \in X$. If $h \in H^n(A)$ and A is a floor for h , then there exists an element h_0 in $H^n(X)$ such that (i) $h_0|_A = h$ and (ii) if F is a closed subset of X such that $h|_F = 0$, then $A \subseteq F$. That is to say, A is the unique floor for h_0 .

Proof. Consider the Mayer-Victoris sequence

$$\rightarrow H^{n-1}(A \cap B) \rightarrow H^n(X) \xrightarrow{J^*} H^n(A) \times H^n(B) \rightarrow H^n(A \cap B) \rightarrow$$

J^* is onto, so there exist an element h_0 in X such that $J^*(h_0) = (h, 0)$; i.e., $h_0|_A = h$ and $h_0|_B = 0$. Suppose that F is a closed subset of X such that $h_0|_F \neq 0$, and consider

$$H^{n-1}(F \cap A \cap B) \rightarrow H^n(F) \xrightarrow{J^*} H^n(F \cap A) \times H^n(F \cap B) \rightarrow$$

J^* is 1-1 and hence $J^*(h_0|_F) \neq 0$. Since $h_0|_{F \cap B} = 0$, we have $h_0|_{F \cap A} \neq 0$. But A is a floor for h and $h_0|_{F \cap A} = h|_{F \cap A}$, so $F \cap A = A$. This completes the proof.

3.2 Remark. Suppose $h \in H^n(X)$ and F is the unique floor for h . Then for $p \in F$ p is an intrinsic point of X . Indeed if V is any open set containing p , then $h|_{X \setminus V} = 0$ because F is unique. Therefore the natural homomorphism $H^n(X) \rightarrow H^n(X \setminus V)$ is not 1-1, and it follows that p is intrinsic. We now have the following corollary.

3.3 Corollary. Suppose S is a continuum with a zero satisfying $S = ESE$. If e is a \mathcal{J} -maximal idempotent of S , then e is not an element of the unique floor for some element of $H^2(S)$.

Proof. Suppose $h \in H^2(S)$, F is a subset of S such that F is the unique floor for h . By theorem 2.4 there exist idempotents g and f in S such that $h|_{Sg \cup Sf} \neq 0$. Since F is unique, F is contained in $Sg \cup Sf$. If $e \in F$, then $e \in Sg \cup Sf$ so it may be assumed that $Se = Sg$ since e is \mathcal{J} -maximal. But then F is contained in $Se \cup Sf$ and F is the unique floor for $h|_{Se \cup Sf}$ in $H^2(Se \cup Sf)$. Now e is the interior of Se relative to $Se \cup Sf$, so by lemma 2.5 e is peripheral in $Se \cup Sf$. This contradicts the preceding remark; hence $e \notin F$ and the proof is complete.

3.4 Lemma. Let R^2 denote the two-dimensional Euclidean plane. Suppose N is a closed totally disconnected subset of R^2 containing a distinguished point p . Then there exists a compact neighborhood M of p satisfying:

- (i) $F(M) \cap N = \square$;
- (ii) $F(M)$ is connected;
- (iii) if C is any connected subset of M containing $F(M)$ and D is a component of $M \setminus D$, then D is homeomorphic to the open unit disk.

Proof. Since p is a component of N , p is a quasi-component; hence there exists an open set U containing p such that U^* is compact and $F(U) \cap N = \square$ [Kuratowski; 1948]. The component V of U containing p satisfied (i) V is open and (ii) $F(V) \subseteq F(U)$; so it may be assumed that U is connected. Let $\{D_n : n \in I\}$ denote the family of components of $R^2 \setminus U^*$. Since U^* is compact there is exactly one unbounded component, call it D_0 , of $R^2 \setminus U^*$. Let $M = U \cup \{D_n : n \in I, D_n \neq D_0\} \cup U^*$. Clearly $R^2 \setminus M = D_0$; and since D_0 is open, M is closed. Moreover since D_0 is the unique unbounded component of $R^2 \setminus U$, M is bounded. Therefore M is a compact neighborhood of p . If $x \in M \setminus M^0$, then $x \notin D_n$ for any component D_n of $R^2 \setminus U^*$ since each D_n is open. Hence $x \in U^*$. But $U \subseteq M^0$ so $x \notin U$ and it follows that $F(M) \subseteq F(U)$. Therefore $F(M) \cap N = \square$ and (i) is verified.

Consider the Mayer-Vietoris sequence,

$$\rightarrow H^0(M) \times H^0(R^2 \setminus M^*) \rightarrow H^0(M \cap (R^2 \setminus M)^*) \rightarrow H^1(R^2) \rightarrow .$$

Since $R^2 \setminus M = D_0$, $(R^2 \setminus M)^*$ is connected. It is clear that M is connected and it follows that $H^0(M \cap (R^2 \setminus M)^*) = 0$; i.e., $F(M)$ is connected. This shows that (ii) holds.

Now suppose that C is a closed connected subset of M containing $F(M)$ and that D is a component of $M \setminus C$. Then D is open relative to M , $D \cap F(M) = \emptyset$, and so D is open. We shall show that $\mathbb{R}^2 \setminus D$ is connected. Suppose Q is some component of $M \setminus D$. If $Q \cap C = \emptyset$, then $Q \subseteq M \setminus C$ and Q is contained in some component D' of $M \setminus C$. Clearly D' is contained in $M \setminus D$ and hence $D' = Q$. But Q is closed in M since $M \setminus D$ is closed, and D' being a component of $M \setminus C$ is open. Therefore Q is open and closed in M , a contradiction. Thus $Q \cap C \neq \emptyset$. Therefore $M \setminus D$ is connected, and it follows that $\mathbb{R}^2 \setminus D$ is connected [Rudin; 1966 p. 262].

We now proceed to the main theorem of this chapter.

3.5 Theorem. Let S be a continuum satisfying $S = ESE$. If e is a \mathcal{J} -maximal idempotent of S and \mathcal{J}_e is totally disconnected, then e does not lie in any two-dimensional Euclidean neighborhood.

Proof. Suppose e is an element of a two-dimensional Euclidean neighborhood V . Since \mathcal{J}_e is totally disconnected, $e \notin K$. Let M be a compact neighborhood of e satisfying the conclusions of lemma 3.4 (N is taken to be $\mathcal{J}_e \cap V$). By remark 0.7 $S \setminus \mathcal{J}_e$ is a maximal ideal of S ; and since $M \cap \mathcal{J}_e = \emptyset$, the ideal $SF(M)S$ is contained in $S \setminus \mathcal{J}_e$. Moreover, $F(M) \subseteq SF(M)S$ since $S = ESE$. Now because $F(M)$ is connected, the set $C = SF(M)S \cap M$ is connected. Letting D denote the component of $M \setminus C$ containing e we have that D is homeomorphic to the open unit disk of \mathbb{R}^2 .

Let $J = SF(M)S$, and consider the Rees quotient modulo J with natural map $\rho : S \rightarrow S/J$. Clearly $\rho(D)$ is homeomorphic to D , so $\rho(D)$ is homeomorphic to the open unit disk in \mathbb{R}^2 . It is

readily verified that $\rho(D)$ is open in S/J and that $\rho(D)^* \setminus \rho(D) = \rho(J)$; i.e., the boundary of $\rho(D)$ in S/J is a point. Therefore $\rho(D)^*$ is homeomorphic to the 2-sphere. The hypotheses of lemma 3.1 are satisfied with $A = \rho(D)^*$, $B = S/J \setminus \rho(D)$, and so $\rho(D)^*$ is the unique floor of some element h_0 in $H^2(S/J)$. But $\rho(E) \in \rho(D)^*$ is a maximal idempotent in S/J , a contradiction to corollary 3.3. Thus the assumption that e belongs to a two-dimensional Euclidean neighborhood is false, and the proof is complete.

Now applying the results of chapter 1, we have the following corollaries.

3.6 Corollary. Let S be a continuum with $S = ESE$, and suppose e is a maximal idempotent in S . If H_e is totally disconnected and e has a 2-dimensional Euclidean neighborhood V , then $S = K$.

Proof. If $S \neq K$, by theorem 1.29 $\mathcal{G}_e \cap V$ is totally disconnected; for otherwise \mathcal{G}_e would be a local separating set of S at some point of $\mathcal{G}_e \cap V$. This contradicts theorem 3.5, and the proof is complete.

3.7 Corollary. Let S be a continuum with $S = ESE$ and $S = SeS$ for some idempotent e in S . If e has a 2-dimensional Euclidean neighborhood V , then $S = K$.

Proof. Since H_e is nowhere dense, $H_e \cap V$ is totally disconnected. Indeed if the component G_0 of $H_e \cap V$ containing e were non-degenerate then G_0 would be a one-dimensional compact abelian group and hence would separate small neighborhoods of e . Again by using the fact that H_e is nowhere dense, this implies that H_e

would be a local separating set of S at some point $H_e \cap V$. This contradicts theorem 1.26, and therefore H_e is totally disconnected. But this contradicts corollary 3.6, and the proof is complete. The following corollary contains a result of Brown [1961], stating that a maximal element in a semilattice on the two must be on the boundary.

3.8 Corollary. Let S be a continuum satisfying $S = ESE$, and suppose e is a maximal element of S . If S is a subset of the plane R^2 , $S \neq K$, then e lies on the boundary of S in R^2 .

Proof. If H_e is totally disconnected, the conclusion follows from corollary 3.7. Otherwise e lies on a circle subgroup G_0 of H_e which separates R^2 . But G_0 does not separate S (theorem 0.11), and the conclusion follows.

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EXAMINATION AND THESIS REPORT

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Major Field: Mathematics

Title of Thesis: Maximal Elements in Compact Semigroups

Approved:

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Date of Examination:

May 14, 1969